1.

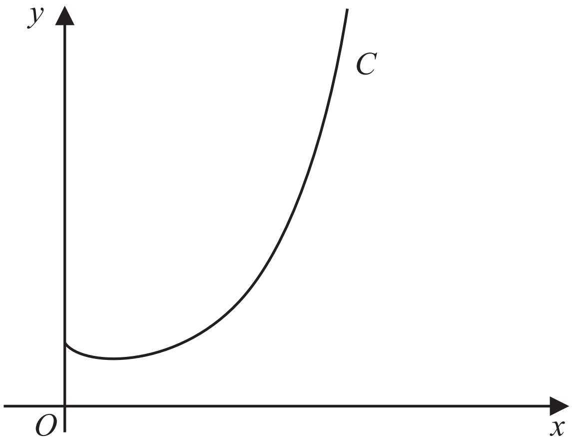


Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x$, $x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

a) $y = x^x$

$\ln y = \ln(x^x)$

$\Rightarrow \ln y = x \cdot \ln(x)$ 1

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln(x) \cdot 1$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln(x)$ 2

$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{y} \cdot 0 = 1 + \ln x$

$\Rightarrow 0 = 1 + \ln x$

$\Rightarrow \ln(x) = -1$

$\Rightarrow e^{\ln x} = e^{-1} \Rightarrow x = \frac{1}{e} = 0.368$ 1

1

Turning Point ?

$\hookrightarrow \frac{dy}{dx} = 0$

log laws : $\ln(a^m) = m \cdot \ln(a)$

Product Rule : $h(x) = f(x) \cdot g(x)$

$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$x \to 1$

$\ln x \to \frac{1}{x}$

b) $y = x^x$

$x = 1.5 \Rightarrow y = 1.5^{1.5} = 1.84$
$x = 1.6 \Rightarrow y = 1.6^{1.6} = 2.12$ (1)

$P(\alpha, 2) \Rightarrow 1.84 < 2 < 2.12$, we also know that C is a continuous curve, hence $\underline{1.5 < \alpha < 1.6}$ (1)

c) $x_{n+1} = 2x_n^{1-x_n}$, $x_1 = 1.5$

$x_2 = 2 \cdot x_1^{1-x_1} = 2 \cdot (1.5)^{1-1.5} = 1.63299..$ (1)
$x_3 = 2 \cdot x_2^{1-x_2} = 1.46626...$
$x_4 = 2 \cdot x_3^{1-x_3} = 1.6731... \Rightarrow x_4 = \underline{1.673}$ (1)

d) $n \to \infty$, what happens to x_n?

- x_n fluctuates between 1 and 2 (1) 1, 2, 1, 2, ...
- x_n will be periodic with period 2 (1)

2. The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = k - \frac{24}{u_n} \qquad u_1 = 2$$

where k is an integer.

Given that $u_1 + 2u_2 + u_3 = 0$

(a) show that

$$3k^2 - 58k + 240 = 0$$

(3)

(b) Find the value of k, giving a reason for your answer.

(2)

(c) Find the value of u_3

(1)

(a) Given $u_1 = 2$

$u_2 = k - \frac{24}{u_1} = k - \frac{24}{2} = k - 12$

$u_3 = k - \frac{24}{u_2} = k - \frac{24}{k-12}$ (1)

Given $u_1 + 2u_2 + u_3 = 0$

$2 + 2(k-12) + k - \frac{24}{k-12} = 0$ (1)

$2 + 2k - 24 + k - \frac{24}{k-12} = 0$

$-22 + 3k - \frac{24}{k-12} = 0$

$(k-12)(-22+3k) - 24 = 0$

$-22k + 3k^2 + 264 - 36k - 24 = 0$

$3k^2 - 58k + 240 = 0$ # (1)

(b) $3k^2 - 58k + 240 = 0$

$(k-6)(3k-40) = 0$

$k = 6, \frac{40}{3}$ (1)

Since k is an integer, $k = 6$ is the answer # (1)

Question continued

$$u_3 = k - \frac{24}{k-12}$$

$$= 6 - \frac{24}{6-12}$$

$$= 6 - \frac{24}{-6}$$

$$= 6 + 4$$

$$= 10 \quad \# \quad 1$$

(Total for Question is 6 marks)

3.

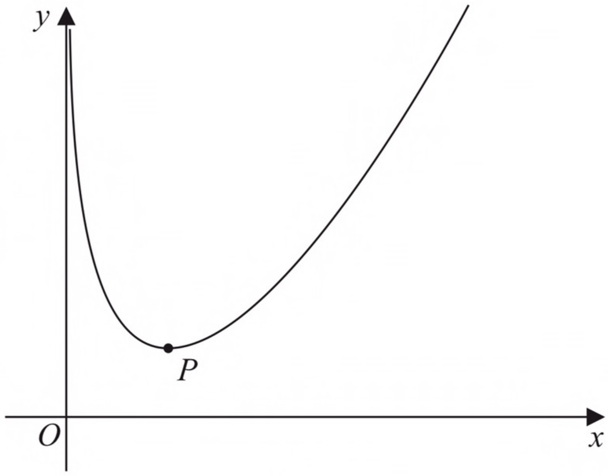


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \qquad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

(4)

a) $y = \underbrace{\frac{4x^2+x}{2\sqrt{x}}} - \underbrace{4\ln x}$, Find $\frac{dy}{dx}$

• Log Differentiation : $\frac{d}{dx}(\ln x) = \frac{1}{x}$

• Quotient Rule : If $h(x) = \frac{f(x)}{g(x)}$

then $h'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$

• $\frac{d}{dx}(4\ln x) = 4\cdot\frac{1}{x} = \boxed{\frac{4}{x}}$ 1

let $h(x) = \frac{4x^2+x}{2\sqrt{x}} \Rightarrow f(x) = 4x^2 + x \rightarrow f'(x) = 8x + 1$

$2\sqrt{x} = 2x^{1/2}$

$g(x) = 2\sqrt{x} \rightarrow g'(x) = \frac{1}{\sqrt{x}}$ 1

$\Rightarrow h'(x) = \frac{(8x+1)(2\sqrt{x}) - (4x^2+x)\left(\frac{1}{\sqrt{x}}\right)}{(2\sqrt{x})^2}$ 1 $= \frac{16x^{3/2} + 2x^{1/2} - \frac{4x^2}{x^{1/2}} - \frac{x}{x^{1/2}}}{4x} = \frac{16x^{3/2} + 2x^{1/2} - 4x^{3/2} - x^{1/2}}{4x}$

$\Rightarrow h'(x) = \frac{12x^{3/2} + x^{1/2}}{4x} = 3x^{1/2} + \frac{1}{4x^{1/2}} = \boxed{3\sqrt{x} + \frac{1}{4\sqrt{x}}}$

$\Rightarrow \frac{dy}{dx} = 3\sqrt{x} + \frac{1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x+1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}} = \frac{dy}{dx}$ as required. 1

The point P, shown in Figure 1, is the minimum turning point on C.

(b) Show that the x coordinate of P is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$$

(3)

b) From part a : $\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$

Our first step is to set $\frac{dy}{dx} = 0$. $\Rightarrow \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}} = 0$

$\Rightarrow 12x^2 + x - 16\sqrt{x} = 0$ $\quad \div \sqrt{x}$

$\Rightarrow 12x^{3/2} + \sqrt{x} - 16 = 0$ $\quad$ 1

$\Rightarrow 12x^{3/2} = 16 - \sqrt{x}$ $\quad \div 12$ 1

$\Rightarrow x^{3/2} = \frac{16}{12} - \frac{\sqrt{x}}{12}$

$\Rightarrow x^{3/2} = \frac{4}{3} - \frac{\sqrt{x}}{12}$

$\Rightarrow x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{2/3}$ as required. 1

(c) Use the iteration formula

$$x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{\frac{2}{3}} \qquad \text{with } x_1 = 2$$

to find (i) the value of x_2 to 5 decimal places,

(ii) the x coordinate of P to 5 decimal places.

(3)

c)

i) $x_1 = 2$ and $x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{2/3} \Rightarrow x_2 = \left(\frac{4}{3} - \frac{\sqrt{x_1}}{12}\right)^{2/3} = \left(\frac{4}{3} - \frac{\sqrt{2}}{12}\right)^{2/3}$ 1

Sub this in!

$x_2 = 1.138935...$

$x_2 = 1.13894$ (5 d.p) 1

ii) $x = 1.15650$ 1

4.

$$f(x) = \ln(2x - 5) + 2x^2 - 30, \quad x > 2.5$$

(a) Show that $f(x) = 0$ has a root α in the interval $[3.5, 4]$

(2)

A student takes 4 as the first approximation to α.

Given $f(4) = 3.099$ and $f'(4) = 16.67$ to 4 significant figures,

(b) apply the Newton-Raphson procedure once to obtain a second approximation for α, giving your answer to 3 significant figures.

(2)

(c) Show that α is the only root of $f(x) = 0$

(2)

a) $f(x) = \ln(2x-5) + 2x^2 - 30$

$f(3.5) = \ln(2\times3.5-5) + 2(3.5)^2 - 30 = -4.81$
$f(4) = 3.10$ 1 $\Rightarrow$ In the interval $[3.5, 4]$ we see a change in sign $\Rightarrow$ there is a root, α, in this interval. 1

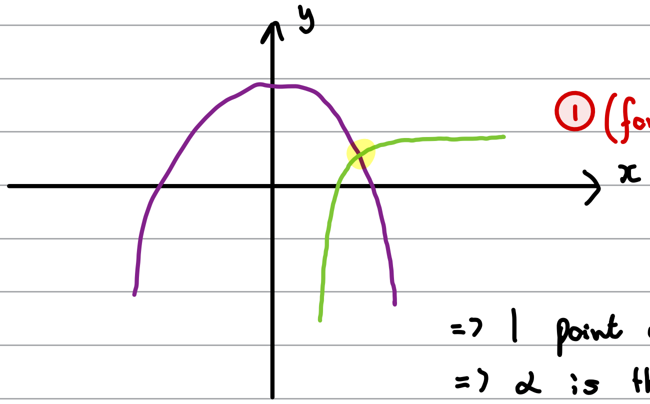
b) Newton-Raphson : $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$

$x_0 = 4$
$f(4) = 3.099$
$f'(4) = 16.67$

$\Rightarrow x_1 = 4 - \dfrac{3.099}{16.67}$ 1 $= 3.81409\ldots \Rightarrow x_1 = \underline{3.81}$ 1

c) $f(x) = 0 \Rightarrow \ln(2x-5) + 2x^2 - 30 = 0$
$\Rightarrow \ln(2x-5) = 30 - 2x^2$

$30 - 2x^2$



1 (for sketch)

$\Rightarrow$ 1 point of intersection
$\Rightarrow \alpha$ is the only root.

5. The equation $2x^3 + x^2 - 1 = 0$ has exactly one real root.

(a) Show that, for this equation, the Newton-Raphson formula can be written

$$x_{n+1} = \frac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$$

(3)

Using the formula given in part (a) with $x_1 = 1$

(b) find the values of x_2 and x_3

(2)

(c) Explain why, for this question, the Newton-Raphson method cannot be used with $x_1 = 0$

(1)

a) Newton-Raphson formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$f(x) = 2x^3 + x^2 - 1$

$f'(x) = 6x^2 + 2x$ ✓

$x_{n+1} = x_n - \frac{2x_n^3 + x_n^2 - 1}{6x_n^2 + 2x_n}$ ✓

$= \frac{x_n(6x_n^2 + 2x_n)}{6x_n^2 + 2x_n} - \left(\frac{2x_n^3 + x_n^2 - 1}{6x_n^2 + 2x_n}\right)$

$= \frac{x_n(6x_n^2 + 2x_n) - 2x_n^3 - x_n^2 + 1}{6x_n^2 + 2x_n}$

$= \frac{6x_n^3 + 2x_n^2 - 2x_n^3 - x_n^2 + 1}{6x_n^2 + 2x_n}$

$x_{n+1} = \frac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$ ✓

Using the formula given in part (a) with $x_1 = 1$

(b) find the values of x_2 and x_3

(2)

b) $x_{n+1} = \dfrac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$

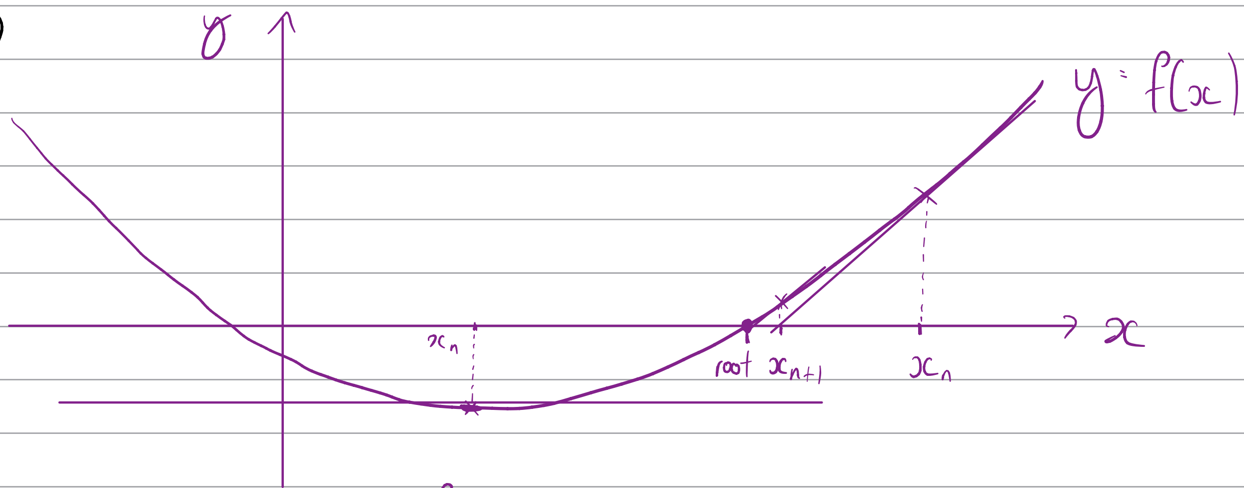
$x_2 = \dfrac{4(1)^3 + (1)^2 + 1}{6(1)^2 + 2(1)} = \dfrac{3}{4}$

$x_3 = \dfrac{4(\frac{3}{4})^3 + (\frac{3}{4})^2 + 1}{6(\frac{3}{4})^2 + 2(\frac{3}{4})} = \dfrac{2}{3}$

(c) Explain why, for this question, the Newton-Raphson method cannot be used with $x_1 = 0$

(1)

c)



$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

at $f'(x_n) = 0$, at a turning point, $\dfrac{f(x_n)}{f'(x_n)}$ is undefined.

For $f(x) = 2x^3 + x^2 - 1$, $f'(x) = 6x^2 + 2x$.
For $x_1 = 0$, $f'(0) = 6(0)^2 + 2(0) = 0$ $\therefore$ a turning point.
Consequently, the tangent will be horizontal, will not meet the x-axis and so not locate the root.

6. The curve with equation $y = 2\ln(8 - x)$ meets the line $y = x$ at a single point, $x = \alpha$.

(a) Show that $3 < \alpha < 4$

(2)

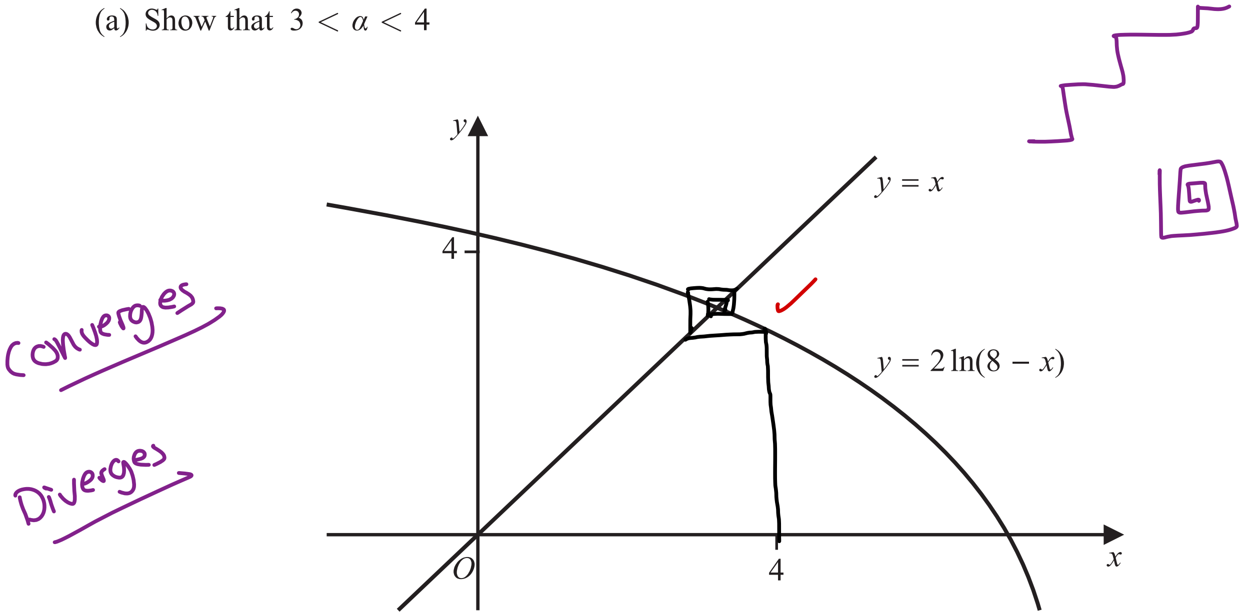


Figure 2

Figure 2 shows the graph of $y = 2\ln(8 - x)$ and the graph of $y = x$.

A student uses the iteration formula

$$x_{n+1} = 2\ln(8 - x_n), \quad n \in \mathbb{N}$$

in an attempt to find an approximation for α.

Using the graph and starting with $x_1 = 4$

(b) determine whether or not this iteration formula can be used to find an approximation for α, justifying your answer.

(2)

a) $2\ln(8-x) = x$

$2\ln(8-x) - x = 0$

$f(x) = 2\ln(8-x) - x$

f(3) & f(4) have a different sign

$f(3) = 2\ln(5) - 3 = 0.22$ (positive)

$f(4) = 2\ln(4) - 4 = -1.23$ (Negative)

f(x) changes sign between 3 & 4, the function is continuous [3, 4] => Root

Question continued

b) can be used to find an approximation for α because the cobweb spirals inwards for cobweb diagram ✓

(see figure 2 for first mark for part b)

(Total for Question is 4 marks)